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AN ALGORITHM FOR FINDING RATIONAL APPROXIMATIONS

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SEPTEMBER 1960

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NOTE

As I was preparing the manuscript of this Report for typing, I discovered, on April 22, 1960, that my method of approximation, at least when restricted to polynomials, is not essentially new. A very similar method is described in [6]. For a comparison of the two methods, see §5.

AN ALGORITHM FOR FINDING RATIONAL APPROXIMATIONS (μ)

H. F. Mattson, Jr.

§1. ABSTRACT

Scientific work frequently requires numerical values of functions. Although much general information is often known about these functions, values of them are nevertheless often difficult to compute; although in almost all cases some method for making this computation, however lengthy it may be, is known. The purpose of a rational approximation to a function is to provide a rapid and convenient way to calculate numerical values of the function to within a predetermined error. This paper considers the question of how to find rational approximations to given functions. In §2 there appear definitions of terms, a precise statement of what our criterion of best fit is, and statements of some classical results. In §3, two closely related iterative methods for finding best rational approximations are defined. In §4, a proof of convergence of these methods is given for a special case (in which both methods are the same). In §5, these methods are compared with some others. In §6, some results obtained by one of the methods of §3 are presented, together with a brief description of the computer program used to obtain them.

§2. DEFINITIONS AND KNOWN RESULTS

A few preliminary definitions are necessary to this discussion. We shall restrict ourselves to the finite interval $I = [a, b]$ on the real line. If we consider the space of all continuous functions (with real values) defined on I, it is natural and commonplace

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to define the distance between two such functions h and g as the maximum absolute value of the difference h - g:

$$\max_{a \leq x \leq b} | h(x) - g(x) | = \| h - g \| .$$

In general, we define $\| G \|$, for any continuous function G defined on I, as

$$\| G \| = \max_{a \leq x \leq b} | G(x) | .$$

Throughout this paper f will denote a fixed, continuous function over I. (We shall impose an additional restriction on f at one point later on.) For given non-negative integers m and n, we consider the family $F_{m,n}$ of all rational functions of the form $S(x) = P(x)/Q(x)$, where P(x) is a polynomial of degree at most m, and Q(x) is a polynomial of degree at most n having no zeros in I. Each S in $F_{m,n}$ is at a certain distance $d_S = \| S - f \| \geq 0$ from f. The set of numbers d_S, with S in $F_{m,n}$ has a greatest lower bound d, a notation which will be fixed throughout this paper. The questions on rational approximation which naturally arise are the following: Is there an R in $F_{m,n}$ such that $d_R = d$? If so, what more can we say about R? In particular, how can we find it?

The answer to the first question is yes: There exists an R in $F_{m,n}$ such that $d_R \leq d_S$ for every S in $F_{m,n}$ [1, p. 53] [1)]. For this R, then, $d_R = d$. Furthermore, such R is unique; that is, if S is any rational function in $F_{m,n}$ different from R, then $d_R < d_S$ [1, p. 56]. R will always denote this best approximation. "Best", or "best-fitting", is here used in the sense previously defined; it is often called "best in the sense of Techebyshev" in the literature.

We now quote two important theorems on rational approximations which will give us more information about R and d. (no pun intended) The first theorem will allow us to find lower bounds for d; the second characterizes R in terms of some properties which will prove to be useful later on.

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1) Numbers in square brackets refer to the bibliography at the end of this Report.

Let R_o be any rational function in $F_{m,n}$ so $R_o(x) = P_o(x)/Q_o(x)$, with $P_o(x) = a_\mu x^{m-\mu} + \ldots + a_m$, $Q_o(x) = b_\nu x^{n-\nu} + \ldots + b_n$, $b_\nu \neq 0$, $0 \leq \mu \leq m$, and $0 \leq \nu \leq n$. Define $N = m + n + 2 - \delta$, where $\delta = \min(\mu, \nu)$.

Assume also that $R_o(x) \neq 0$ and that $P_o(x)$ and $Q_o(x)$ have no common divisor.

THEOREM A. Suppose that the error function $E_o = R_o - f$ at some N points $X_1 < \ldots < x_N$ in I assumes respectively the values $-v_1, +v_2, -v_3, \ldots, (-1)^N v_N$ different from zero and of alternating signs (thus all v's have the same sign).

If R_1 is any member of $F_{m,n}$ with error function $E_1 = R_1 - f$, then

$$d_{R_1} = \|E_1\| \geq \min \{|v_1|, \ldots, |v_N|\} .$$

(If $P_o = 0$, then the same inequality holds with $N = n + 2$.)

A consequence of this theorem is that $d = d_R \geq \min \{|v_1|, \ldots, |v_N|\}$. And if R_1 is any member of $F_{m,n}$ the error function of which has sufficiently many extreme values of alternating signs, then d is not less than the minimum of these extreme values (in magnitude).

THEOREM B. R_o is the best-fitting rational function R if and only if there exists at least N points $x_1 < \ldots < x_N$ in I at which $E_o = R_o - f$ assumes the values $E_o(x_i) = (-1)^{i+\alpha} \|E_o\|$, $i = 1, \ldots, N$, where α is either 0 for all i or 1 for all i.

(R = 0 is the best-fitting rational function in $F_{m,n}$ if, and only if, there are at least $N = n + 2$ points $x_1 < \ldots < x_N$ for which $f(x_i) = (-1)^{i+\alpha} \|f\|$.)

For the proof of Theorem A, see [1, pp. 52-53]. A proof of Theorem B also occurs in [1, pp. 55-57].

In proving the uniqueness of R [1, pp. 56-57], mentioned earlier, one uses Theorem B.

We shall use the terms "extremum" and extreme value" as follows: An extremum of the function g is a point in I at which g takes an extreme value. If $R_o \in F_{m,n}$ we define an admissible set of extrema of E_o ($E_o = R_o - f$) to be N points $x_1 < \ldots < x_N$ in I such that

1) each x_i is an extremum of E_o, and

2) $E_o(x_i) \neq 0$ and alternates in sign as i increases from 1 to N.

The set $\{E_o(x_i) ; i = 1, \ldots, N\}$ is then called an admissible set of extreme values of E_o.

Finally, let me observe that the only extreme (of any function G) pertinent to the situation under discussion in this Report are the maxima of $|G|$. Therefore "extrema" should be tacitly so understood here.

§ 3. DEFINITIONS OF ALGORITHMS

We now define two iterative procedures which in some cases (see § 6) are known to converge to the best-fitting rational approximation to our given continuous function f over $I = [a, b]$. (There are no cases known to me in which the procedures do not converge to f, but a proof of convergence is known to me only for the restriction to $n = 0$, $f \in C^n$.)

We shall call our first algorithm the "non-linear" algorithm and our second the "linear" algorithm. Both have in common the following first step:

1.° For a given m, n select (say, by interpolation at the estimated zeros of E) $R_o \in F_{m,n}$ such that $E_o = R_o - f$ has an admissible set of extrema.

Nonlinear Method:

2.° Given $R_{j-1} \in F_{m,n}$, such that $E_{j-1} = R_{j-1} - f$ has an admissible set of extreme $x_1, \ldots, x_N$, determine $R_j \in F_{m,n}$ by imposing the N conditions

$R_j(x_i) - f(x_i) = (-1)^i y_j$ $i = 1, \ldots, N$, where y_j is an unknown. The other unknowns are, of course, the coefficients occurring in P_j and Q_j in R_j.[2)]

Linear Method:

2^o. Given $R_{j-1} = P_{j-1}/Q_{j-1}$ $\quad F_{m,n}$ $j \geq 0$, such that $E_{j-1} = R_{j-1} - f$ has an admissible set of extrema $x_1 < \ldots < x_N$, determine $R_j = P_j/Q_j \in F_{m,n}$ by the N conditions

$$P_j(x_i) - f(x_i)\, Q_j(x_i) = (-1)^i Q_{j-1}(x_i)\, y_j, \quad i = 1, \ldots, N.$$

Choose the leading coefficient of each Q_j to be 1. Notice that these equations are linear in the unknowns y_j and the coefficients occurring in P_j and Q_j.[2)]

Comments on these methods:

The motivation for the non-linear method clearly comes from the characterization of the best-fitting rational function in Theorem B. The desired best-fitting rational function is obviously a fixed point of the transformation defined in Step 2^o. (modulo the complications mentioned in footnote 2). The linear method is derived from the non-linear one in an obvious way, and it also has the desired best-fitting rational function as a fixed point (modulo the complications mentioned in footnote 2).

It is not known whether either of these methods actually stays inside $F_{m,n}$ in general. That is, for some j, R_j as defined in either method may, a priori, fail to exist. Also, for some j, R_j may, a priori, have a pole in I. If one satisfied these pre-conditions, the central question would remain, namely, whether one or the other method converges. These three questions appear to be difficult for the general case

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2) It may happen that $R_j - f$ has more than N extrema. In such a case I require the choice of a particular admissible set of extrema. I have stated the method in its simplest form here, for clarity; it appears in full generality as step $2'^o$ in §4.

(i.e., m and n arbitrary, f any continuous function on I) of the non-linear method and more difficult for that of the linear method. But in the case n = 0 both methods coincide, and a considerable number of the necessary results can be proved for arbitrary f. Finally, if one then restricts f to have m continuous derivatives, a full existence and convergence proof is possible. We present this proof in the next section.

§ 4. PROOFS FOR THE CASE n = 0

We now confine ourselves to polynomial approximations P_j to our given f. Both methods are the same in this case. Our first step is to satisfy the pre-conditions by proving the existence of P_{j+1}, given a P_j having the required properties. The proof leads naturally to further results, all of which we include in the following theorem.

THEOREM 1. Let P_o be a polynomial, either 0 or degree $m - \mu$, where $0 \leq \mu \leq m$, such that there are $N = m + 2$ points

$$(a \leq)\ x_1 < \ldots < x_N\ (\leq b)$$

at which the error function $E_o = P_o - f$ takes on respectively the non-zero values $E_o(x_i) = (-1)^i v_i$ $i = 1, \ldots, N$, where all v_i have the same sign. Then there exists a polynomial P_1 of degree at most m, and a number y such that the error function $E_1 = P_1 - f$ takes the values

$$E_1(x_i) = (-1)^i y \quad (*),$$

and

$$\left.\begin{array}{l} \text{a) } y \text{ is uniquely determined by the condition } (*), \\ \text{b) If not all } v_i \text{ are equal, then} \\ \qquad \min_{1 \leq i \leq N} |v_i| < |y| < \max_{1 \leq i \leq N} |v_i| \text{ (therefore } y \neq 0), \\ \text{c) Sign } y = \text{sign } \{v_i\}\ . \end{array}\right\} \qquad (4.1)$$

PROOF. We are given N distinct points x_i at which there hold the equations

$$P_o(x_i) - [f(x_i) + (-1)^i v_i] = 0, \quad i = 1, \ldots, N.$$

These equations may be thought of as N homogeneous equations, already solved, for the N-1 coefficients of an m-th degree polynomial, plus one more "unknown". That is, $(0, \ldots, 0, a_\mu, a_{1+\mu} \ldots, a_m, 1)$ is the solution-vector (transposed), and the N x N coefficient-matrix is

$$q \qquad M = \begin{bmatrix} x_1^m & x_1^{m-1} & \ldots & 1 & -[f(x_1) - v_1] \\ x_2^m & x_2^{m-1} & \ldots & 1 & -[f(x_2) + v_2] \\ \cdot & & & & \\ \cdot & & & & \\ \cdot & & & & \\ x_N^m & x_N^{m-1} & \ldots & 1 & -[f(x_N) + v_N] \end{bmatrix}.$$

The existence of P_o with the given properties imply that the determinant of the matrix M is zero.

What we first wish to prove is that there is a number y such that the matrix we obtain by replacing each v_i by y in M is also zero. (Such a y would imply the existence of the desired P_1.) To this end we expand det (M) by cofactors of the last column, obtaining

$$\sum_{i=1}^{N} (-1)^{N+i} e_i [f(x_i) + (-1)^i v_i] = 0 \qquad (4.2);$$

where the minor e_i is the van der Monde determinant

$$e_i = \begin{vmatrix} x_1^m & x_2^{m-1} & \cdots & 1 \\ \vdots & & & \\ \hline x_1^m & x_i^{m-1} & \cdots & 1 \\ \hline \vdots & & & \\ x_N^m & x_N^{m-1} & \cdots & 1 \end{vmatrix} \qquad \text{(this row omitted)} ;$$

thus we have, by the well known formula,

$$e_i = \prod_{\substack{j<k \\ j,\, k \neq i}} (x_j - x_k) .$$

Since the x_i's are all distinct, each $e_i \neq 0$; since the x_i's are arranged in increasing order, all e_i have the same sign, each one being the product of the same number of negative factors. Having noted these properties of the e_i's, we now rewrite equation (4.2) as

$$e_1 v_1 + \ldots + e_N v_N = - \sum_{i=1}^{N} (-1)^i e_i f(x_i) = {}_{df} e. \qquad (4.3)$$

This shows that $|e| = \sum |e_i v_i| > 0$ and that sign e = (sign e_1) (sign v_1). We can now immediately satisfy our requirements for the existence of P_1 by choosing y to satisfy

$$e_1 y + \ldots + e_N y = e, \qquad (4.4)$$

of $y \in e / \sum_{i=1}^{N} e_i$. Furthermore, this is the only choice open to us for y.

A comparison of (4.2) and (4.3) shows that min $|v_i| < |y| < \max |v_i|$ unless all v_i are equal (in which case $y = v_i$); and, finally, sign y = sign e. sign e_i = sign v_i. QED.

There are two points to notice about this theorem. One is that we do not require the x_i's to be extrema of E_o but only to be points where E_o alternates in sign. In the application of this theorem to our iterative process, however, we shall take them as extrema.

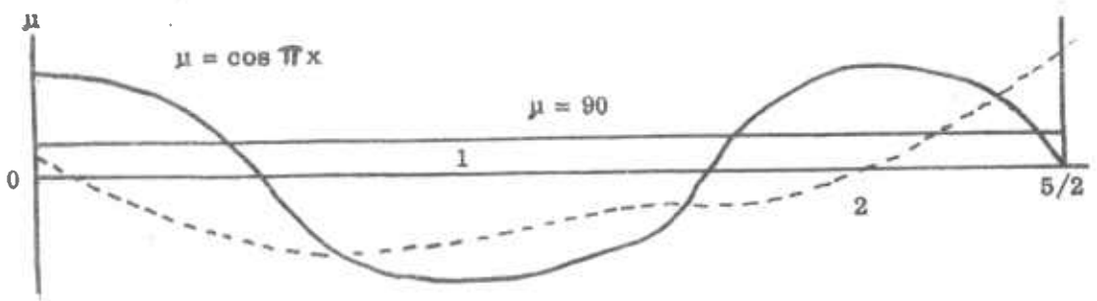
The other point is that although P_o may have degree less than m, the same is not necessarily true of P_1, as the following example shows:

Let $f(x) = \cos \pi x$, over $I = [a, b] = [0, 5/2]$. Take $m = 2$, so that $N = 4$. For P_o, take $P_o(x) = a_o$, with $0 < a_o < 1$. Then we have

$$x_1 = 0 \qquad x_2 = 1 \qquad x_3 = 2 \qquad x_4 = 5/2$$

$$v_1 = 1-a_o \quad v_2 = 1+a_o \quad v_3 = 1-a_o \quad v_4 = a_o \ ,$$

as is obvious from the following sketch:



This same sketch makes it obvious that there is no straight line $u = P_1(x) = a'_o x + a'_1$ having deviations of equal magnitude at the x_i's. Therefore P_1 will be a parabola, something like the one sketched with a dashed line.

What we have proved in Theorem 1 is that we can always construct our sequence $\{P_j\}$ satisfying the conditions of step 2^o, provided there exists a first polynomial P_o satisfying the condition of step 1^o. But we can always find P_o by solving $P_o(x_i) - f(x_i) = (-1)^i y_o$ for the unknown coefficients in P_o and the unknown y_o, for any distinct $x_1, \ldots, x_N \in I$.[3] The existence of this P_o and y_o is given by the proof of Theorem 1, in which y_o is given by (4.4) with e defined by (4.3). It is only necessary to choose the x_i so that $e \neq 0$, the possibility of which follows, in case f is not a polynomial of degree at most m, from the existence of the best-fitting polynomial and the consequent existence of v_i's with alternating signs.

Having shown the existence of our sequence $\{P_j\}$, we now show that, under suitable restrictions, it converges to P.

We first must modify our definition of step 2^o to take account of the possibility that there are more than one set of admissible extrema. We prove a simple lemma:

LEMMA 1. If $E_j = P_j - f$ has an admissible set of extrema, then there is an admissible set of extrema values containing $+\|P_j - f\| = \pm d_j$.

PROOF. Let $\pm d_j$ occur as a value of E_j at x. If x is already in the admissible set of extrema, then we are done. If x lies between two of the extrema x' and x'', then $E_j(x)$ must have the same sign as one of $E_j(x')$ and $E_j(x'')$. Replace that one by $E_j(x)$. If x lies entirely to the left of the admissible extrema, then either replace the one nearest x by x, or delete the farthest one and include x, all depending on whether or not the sign of E_j at the one nearest x is the same as that of $E_j(x)$. QED.

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3) I am indebted to Novodvorskii and Pinsker, [4], via Shenitzer [6], for this point. It is slightly easier to prove the possibility of this than to show the possibility of avoiding a tangency of P_o and f when interpolating to f at the estimated zeros of E, which I had suggested earlier. In order to obtain linearity of the equations for R_o when $n > 0$, however, one needs to interpolate as first suggested.

It follows immediately from this lemma that if we choose a set of admissible extreme values having the largest possible minimum (in magnitude), then we may replace it by a set containing $\pm \| E_j \|$ without changing the magnitude of the minimum extreme value in it. It is this procedure that we follow when there is more than one choice of a set of admissible extreme values at any stage. Accordingly, we substitute the following step $2'^{o}$ for step 2^{o};

$2'^{o}$. Given $P_j \in F_{m,o}$ such that $P_j - f$ has a set of (N) admissible extreme values, choose a set S of admissible extreme values containing the largest possible minimum (in magnitude) and containing $\pm \| P_j - f \|$. Let $x_1, \ldots, x_N$ be the extrema corresponding to this set S (i.e., $S = \{ E_j(x_i) ;\ i = 1, \ldots, N \}$). Determine $P_{j+1}(x) = a_o x^m + \ldots + a_m$ by imposing the N conditions

$$P_{j+1}(x_i) - f(x_i) = (-1)^i y_{j+1}, \quad i = 1, \ldots, N,$$

where y_{j+1} is an unknown.

The possibility of carrying out step $2'^{o}$ has been proved in Theorem 1 and Lemma 1.

We now turn to the proof of convergence.

LEMMA 2. The sequence $\{ | y_j | \}$ determined by step $2'^{o}$ is strictly increasing and bounded above by $d = \| E - f \|$; (unless some $P_j = P$; then all $| y_k | = d$ for $k \geq j$).

PROOF. We are given that $| y_j | = | E_j(x_i) |$, where the x_i are the extrema belonging to the set S of extreme values of E_{j-1} defined in step $2'^{o}$. Let $V_i = E_{j-1}(x_i)$. Let $S' = \{ (-1)^{\alpha + i} v'_i \}$ be a set of extreme values of E_j as prescribed by step $2'^{o}$. Since there is an admissible set of extreme values of E_j such that $| y_j |$ is smaller (in magnitude) than all of them, it follows that

$$| y_j | \leq \min | v_i | . \tag{4.5}$$

Now (4.5) plus Theorem 1 yields $| y_j | \leq \min | v_i' | < | y_{j+1} |$. And finally, Theorem A gives $| y_j | \leq \min | v_i | \leq ||E||$ = d. QED.

We now prove a convergence theorem.

THEOREM 2. Every convergent subsequence of $\{P_j\}$ has limit P.

PROOF. Every subsequence $\{P_{j_k}\}$, convergent or not, has the property that the sequence of attached $| y_{j_k} |$ converges to some limit $d' \leq d$. Let P* be the limit $\lim_{k \to \infty} P_{j_k}$ of our convergent subsequence. Then P* is a polynomial of degree at most m, (because the sequences of coefficients of the P_{j_k} are bounded; therefore there is a subsequence of the subsequence P_{j_k} which has a polynomial p(x) as limit. But since the subsequence P_{j_k} is already convergent, the limit P* must be the polynomial p(x) just mentioned. Finally, the coefficients of the P_{j_k} are bounded because the polynomials are bounded at m + 1, in fact at all, points of I.) Thus P* is the uniform limit of $\{P_{j_k}\}$. We shall prove that $d' = ||P^* - f||$. First, $EI = P^* - f$ has an admissible set S of extreme values, since P* is the uniform limit of $\{P_{j_k}\}$; and, for the same reason, the set S has d' as minimum magnitude, in view of the inequalities (4.1) of Theorem 1. Therefore we may, and do, apply step 2'° to P*, obtaining P**. If $||E^*|| > d'$, then the value of the y* found in step 2'° (satisfying E** (extrema of E*) = ± y*) would satisfy $(y^*) > d'$. We choose k large enough so that corresponding admissible extrema of E_{j_k} and E* are close enough to each other to yield $| y_{j_k} |$ so close to $| y^* |$ that $|y_{j_k}| > d'$. This is possible since y, determined by formula (4.4), depends continuously on the extrema. But this result contradicts the definition of d'. Therefore $||E^*|| = d' \leq d$. But since we always have $||E^*|| \geq d$, we have proved that d = d'. By the uniqueness of P, we conclude that P* = P. QED.

Now we are almost finished, for if we knew that $\{P_j\}$ were a bounded sequence, we could conclude from the previous theorem that $\{P_{j_i}\}$ converged to P.[4] We now introduce a hypothesis, probably stronger than necessary, which implies the boundedness of $\{P_j\}$.

THEOREM 3. If f has m continuous derivatives, then the sequence $\{P_j\}$ constructed according to either step 2^o or step $2'^o$ is bounded over I.

PROOF. Each error function $E_j = P_j - f$ has at least $N - 2 = m$ distinct extrema interior to I. The existence of a continuous first derivative implies that E'_j has at least m distinct zeros inside I. Therefore E'_j has at least m-1 distinct extrema and E''_j an equal number of zeros inside I. Continuing in this way, we find that $E_j^{(m)}$ has at least one zero inside I. Therefore we have $P_j^{(m)}(z') = m!\, a_{oj} = f^{(m)}(z')$ for some z' in I, where a_{oj} is the leading coefficient of P_j. Since $f^{(m)}$ is assumed continuous on I, it is founded there, from which it follows that $\{P_j^{(m)}\}$ is bounded.

From the relations

$$P_j^{(\alpha-1)}(x) = \int_z^x P_j^{(\alpha)}(t)\, dt + f^{(\alpha-1)}(z), \quad \alpha = m, m-1, \ldots, 1,$$

where z is a zero of $E^{(\alpha-1)}$, we conclude inductively that $\{P_j\}$ is bounded, since all derivatives of f are bounded, for order not greater than m.

Incidentally, we could conclude easily from the above proof that the coefficients of the P_j are all bounded: $\{a_{oj}\}$ is bounded since $P_j^{(m)} = m!\, a_{oj}$ is bounded; one integration introduces the a_{1j}, which are therefore bounded, and so on. But we already know the boundedness of the coefficients follows from that of the polynomials, in general.

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4) We would have: In a complete matric space (here the Euclidean (m + 1)-space of coefficients) a bounded sequence with at most one limit point converges.

We summarize the import of this section in the following theorem.

THEOREM 5. Let $f \in C^m$ on I, and let the sequence $\{P_j\}$ of polynomials be defined by the rules: P_0 -f has an admissible set of extrema; each P_j, $j > 0$, is obtained from P_{j-1} by the rule of step 2'°. Then $\{P_j\}$ converges uniformly on I to P, the polynomial of degree at most m which lies nearest to f in the matric defined in § 2 .

§ 5. COMPARISONS WITH OTHER METHODS

The methods of Remez [5], represented in some fashion in [4], is described in English in 6 for approximations of the form s(x) P(x), where s is a fixed continuous function with no zeros in I, and P is a polynomial. Proofs, said to be given in [4], are omitted from [6]. The latter reference is the only one I have been able to read to date.

We now describe Remez's method in our terminology:

The method begins by the choice of the initial approximation P_0 as described in § 4: For $x_1 < \ldots < x_N \in I$, we determine P_0 by the N equations $P_0(x_i) - f(x_i) = (-1)^i y_0$. Step 2. Let x' be a point of I at which $E_0 = P_0 - f$ takes the value $\pm ||E_0||$. Replace <u>one</u> of the x_i by x', calling the resulting points $x_{11} < x_{21} < \ldots < x_{N1}$, in such a way that they are an admissible set of extrema of E_0 (Cf. Lemma 1.) Now determine P_1 by solving the N equations $P_1(x_{i1}) - f(x_{i1}) = (-1)^i y_1$, i = 1, ..., N. Find P_2 by replacing one of the x_{i1} by x'' such that $E_1(x'') = \pm ||E_1||$, and so on.

It is obvious that the conclusions of Theorem 1 hold here also, and that the y_j's of Remez increase monotonically (in magnitude) to d. The rest of our proof of convergence clearly carries over to this process without essential change. The presence of the function s would complicate the proofs in no essential way; s was omitted from the present report chiefly for reasons of clarity.

As a practical matter, the present method should converge faster than that of Remez, since the $|y_j|$ are larger than the corresponding quantities in Remez's method. This quicker convergence is paid for by a more complicated choice of the admissible extrema x_{i_j} at each stage, however, except for the case when there are always exactly N admissible extrema. In this important special case, there is no more difficulty in carrying out the present method than there is in doing that for Remez's; For in order to find $||E_j||$ one must find all extreme values of E_j.

If the computation of f(x) were difficult, then the method of the present report might well be preferable to that of Remez.

I was led to the method of this report partly by ruminating on Hastings's method, [2], as defined by P. W. Ketchum in Mathematical Reviews [3] (Hastings's book [2] suffers from a certain lack of definition). For me, the central point of both Hastings's method and my own is the "iterative assumption" of stability of the extrema of E_j. This point plus Theorem B led me to my method. I then noticed the following comparison: Hastings linearizes the N non-linear equations in step 2^o by assigning a numerical value to y_j, thus reducing the number of unknowns to N-1; he solves N-1 equations and hopes for a correct value at the N-th extrema. I linearize these equations by replacing the unknown coefficient of y_j by a known number, thus preserving the number of unknowns.

§6. EXPERIMENTAL DATA

The so-called "linear algorithm" of §3 for finding rational approximations has been programmed, with m = n = 2, for the Cambridge Computer. The following are the functions f approximated, the interval I, the value of $||E_k||$ obtained, the smallest maximum value of $|E_k|$, and the number k of times Step 2^o was performed in order to obtain the final approximation:[5)]

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5) I wish to acknowledge most gratefully the kind assistance of Miss Helen Willett in obtaining these results from the Cambridge Computer.

f	I	$\|\|E_k\|\|$	min max $\|E_k\|$	k
exp	$[-1, 1]$	8.71×10^{-5}	8.66×10^{-5}	4
log	$[1, 2]$	1.75×10^{-6}	1.68×10^{-6}	5
sine	$[0.6, 7.0]$	0.266	0.263	13

In each of the above three "experiments", there were exactly $6 = N$ extrema of E_j for each j; the extreme values always alternated properly in sign. These extrema were maxima of $|E_j|$. Two extrema were always at the end-points of I.

The minimum magnitude of the extreme values is shown in order to provide an idea of the closeness of the last approximation R_k to the best approximation R. In the cases exp and log, these numbers and those under the heading $||E||$ are probably not accurate to the three significant figures presented, since they are the last three significant figures of the eight available on the Cambridge Computer.

There follow the values of the coefficients of the approximations R_k discussed above. In each case R_k has the form

$$R_k(x) = (a_2 x^2 + a_1 x + a_0) / (x^2 + b_1 + b_0).$$

	a_2	a_1	a_0	b_1	b_0
exp	1.1045366	6.5455741	12.869739	-6.3197327	12.868806
log	3.3615004	1.0750524	-5.3365379	5.6992101	1.9999023
sine	.88630520	-8.4079627	18.363517	-9.0106829	21.528251

Some tests of significance of digits in the coefficients of the approximation to exp were made. It was found that rounding the coefficients to seven significant figures produced a spread of 0.25×10^{-5} between minimum and maximum extreme values, whereas rounding to six decimal places produced a spread of 0.03×10^{-5}, the maxi-

mum being 8.701×10^{-5}, the minimum 8.67×10^{-5}. These numbers of course are potentially in error because of round-off in the machine, but they indicate that the last-mentioned rounding produces a result just as good as the original.

A brief description of the program follows. Letting h denote (b - a)/20, the computer finds R_0 by interpolating to f at the five equally spaced points $x_1 < \ldots < x_5$, where $x_1 = a + h$ and $x_5 = b - h$. The extrema of E_0 are then found: At every stage the program assumes that a and b are extrema of E_j; the interior extrema are found by solving $E_j'(x) = 0$ by the method of regula falsi. The equations of step 2° are then solved; and the process is repeated. The criterion for stopping at j = k is that the extrema of E_k be not too different from the corresponding extrema of E_{k-1}, which are stored at eaoh stage. The coefficients a_i and b_i of each R_j are printed, and at the end the values of the extrema of E_k and the corresponding extreme values are printed.

The arbitrariness of the above procedure for finding R_0 can lead to difficulty. In particular, for f = sine, it gave an R_0 having poles in I for I = [0.1, 6.8] and I = [0.5, 6.9]. Up to now, the method has "converged", however, so long as R_0 had no poles in I. As now programmed, however, it would probably fail if some E_j had more than N = 6 extrema, or if a (or b) were a minimum of $|E_j|$ rather than a maximum.

At no point, except in the non-essential final print-out, is it necessary in this method to compute values of E_j.

APPENDIX

(Added "in Proof")

Now that I have seen the paper [4] in translation, let me describe it a little more fully. This paper proves the convergence of a process similar to but more general than my own polynomial algorithm. Specifically, the Russian authors consider a class Ω of functions Δ which can be thought of as a generalization of $\{P - f\}$ for a given continuous f an all polynomials P of degree at most m. The sequence $\Delta_0, \Delta_1, \ldots$ is constructed by equating the values of Δ_k to $(-1)^i y_k$ at <u>any</u> $n + 2$ points satisfying certain properties which are more general than those which I required.

The proofs in [4] are quite different in execution than mine, but the general similarity of direction is readily apparent.

It appears that the method of [4] does not apply to the class of rational approximations which I discussed.

BIBLIOGRAPHY

1. ACHIEZER, N. I., The Theory of Approximation, Frederick U
   1956.

2. HASTINGS, CECIL, JR., Approximations for Digital Computers
   University Press, 1955.

3. KETCHUM, P. W., (Review of [3]), Math. Reviews, vol. 16

4. NOVODVORSKII, E. N. and PUISKER, I. SH., "On a Process o
   Maxima", Uspehi Mat. Nauk., vol. 6, Issue 6 (46), 1951, pp.
   (Available in a translation by A. Schenitzer from the Library
   of Mathematical Sciences, New York University, New York 3,

5. REMEZ, YA. L., "On a Method of Chebychev type Approximati
   Ukr. AN. 1935.

6. SHENITZER, A., Chebychev Approximation of a Continuous Fun
   of Functions", J. Assoc. Comput. Mach. 4 (1956), pp. 30-35

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Errata

Page	Line	
		A minus sign prefixed to a line number indicates the count from the bottom of the page.
1	(Title)	delete (U)
1	-1	for September 1, 1960 read 25 April 1960
2	-6	for Techebyshev read Tschebyscheff
3	2	for all ν read ν (three places)
3	3	for both ν read ν
3	6	for X_1 read x_1
3	-8	for exists read exist
4	9	for extreme read extrema
4	-1	for extreme read extrema
5	4	before $F_{m,n}$ insert ϵ
5	7	for $(-1)^f$ read $(-1)^i$
6	11	for or degree $m-\mu$ read or of degree $m-\mu$
7	7	delete q
8	2	for x_2 read x_1
8	3	for x_1 read x_i
8	-1	for of $y \in$ read or $y \in$
9	sketch	for all μ read u; for 90 read a_o. Also, for the higher plane curve sketched with the dashed line, please imagine a parabola with vertex at about (1, -1/3) and passing through (0, 1/3) and (5/2, 2/3)
11	7	for (N) read N
11	-6	for $E_j(x_i)$ read $\|E_j(x_i)\|$
11	-5	for V_i read v_i
11	-1	for v_i read v_i'
12	1	for $\|y_{j+1}$ read $\|v_{j+1}\|$
12	2	for v_i read v_i'
12	13	for EI read E*

Errata

Page	Line	
12	-6	for (y*) read $\|y^*\|$
13	13	for founded read bou
15	-4	for a_{1j} · read $a_{1j'}$
13	-2	for matric read metr
14	5	for matric read metr
14	9	for in 6 for read in {
15	7	for For read for
18	5	for f an all read f a
19	4	for Review of [3] re
19	8	for PUISKER read Pu
19	8	for Process or rea